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1994 J. Phys. A: Math. Gen. 27 L107

(http://iopscience.iop.org/0305-4470/27/5/001)

J. Phys. A: Math. Gen. 27 (1994) L107–L112. Printed in the UK

LETTER TO THE EDITOR

Fractal measures of mean first passage time in the presence of Sinai disorder

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Received 10 November 1993

Abstract. We consider mean first passage time (MFPT) of random walks from one end to the other of a segment of a Sinai lattice, with a reflecting left boundary and an absorbing right boundary. Random fields are located at each site and can accept values $\frac{1}{2} \pm \epsilon$ with equal probability. We investigate the nature of the distribution of MFPT over Sinai fields, employing multifractal formalisms. We calculate the fractal dimension $D(0)$ and find it varies nearly linearly with ϵ, the strength of Sinai disorder. We then study the scaling behaviour of the partition function. To this end we make a scaling ansatz and fit it to our exact results on finite lattices, which yields the scaling exponents $\tau(q)$. We report results on the scaling exponents for various values of the disorder parameter.

Systems with quenched-in disorder arise in a variety of contexts in condensed matter physics [1]. The transport properties of these systems exhibit interesting anomalous features which are not usually tractable by analytical means. Hence one focuses attention on simple prototypes and investigates the transport properties employing random walk models.

The Sinai model [2] proposed in 1982 is a classic example. It consists of a particle executing a random walk on a one-dimensional lattice. At each lattice site i, the right and left jump probabilities are denoted by p_i and $1 - p_i$ respectively. The set $\{p_i\}$ constitutes identically distributed independent random variables with a common distribution prescribed in such a way that the logarithm of the quantity $\beta = (1 - p)/p$ has zero mean and finite variance, σ^2. The mean-square displacement of the particle increases ultra-slowly as the fourth power of logarithm of time. This anomalous behaviour is not difficult to understand, if we realise that over a typical distance N, a potential barrier proportional to $\sqrt{N}$ develops by virtue of the addition of random biases with zero mean. To overcome such an Arrhenius barrier, it requires typically a time of the order of $\exp[\sqrt{N}]$, from which it immediately follows that the mean square displacement should go like $[\ln(t)]$ [4]. It should be noted, however that the anomalous diffusion characterizes the typical behaviour.

In the context of random walk models, there is an alternate way of investigating the transport properties of systems which, many-a-time prove simpler. It consists of calculating the time taken by the particle to go from one end to the other, of a finite segment of the lattice. The advantages of such first passage time (FPT) formulations [3–8] for studying

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transport properties of systems with quenched-in disorder arise from the fact that the lattice is finite and hence different disorder configurations can be enumerated exhaustively and exactly.

The FPT formulation is usually made as follows. We consider random walks on a segment of a one-dimensional lattice, with a reflecting left boundary at site $i = 0$, and an absorbing right boundary at site $i = N$. The random walk starts at 0 and jumps to site 1 with probability p_0 or stays at the 0 itself with probability $1 - p_0$. Eventually the random walk gets absorbed at N. Let $t_{0,N}$ denote the number of steps the random walk takes to reach the absorbing site, starting from site 0. The random variable $t_{0,N}$ is called the FPT. We consider a simple dichotomic model of the Sinai disorder. We prescribe p to take only two values, $\frac{1}{2} \pm \epsilon$ with equal probability, where $0 < \epsilon < 1/2$ is a parameter that measures the strength of the disorder. It is easily verified that the above prescription obeys the Sinai conditions. Let t denote the mean first passage time (MFPT). In other words $t = \langle t_{0,N} \rangle$, where the angular brackets denote averaging over all possible random walks on a given realization of the random lattice. It has been shown [3–5] that the MFPT averaged over Sinai disorder diverges with system size as $\exp[\gamma N]$ where $\gamma = \ln[\langle \beta \rangle]$, while the typical value of MFPT diverges comparatively slowly as $\exp[\sigma\sqrt{N}]$. This implies that fluctuations of MFPT increase when system size N increases. The central limit theorem or its equivalent is not applicable for such problems. In fact it has been shown that the distribution $\rho(t)$ of the MFPT over Sinai disorder has a $1/t$ tail [6–7]. In order to obtain a further and better characterization of diffusion on a Sinai lattice, we have studied the multifractal measures of the MFPT as described below.

Our starting point is the analytical expression derived for the MFPT for a given realization of the random lattice. Details of the derivation can be found elsewhere [4], and we give below the expression for t, explicitly in terms of $p_0, p_1, \ldots p_{N-1}$.

$$t = \sum_{k=0}^{N-1} \frac{1}{p_k} + \sum_{k=0}^{N-2} \frac{1}{p_k} \sum_{i=k+1}^{N-1} \prod_{j=k+1}^{i} \frac{1 - p_j}{p_j}. \tag{1}$$

Formal expressions for mean and typical values of t can be obtained [4–5] from this expression. In this letter, however, we shall be interested in the multifractal characterization of the distribution $\rho(t)$.

For a chain of size N and the dichotomic model of Sinai disorder, there are 2^N realizations of the random lattice possible. We enumerate exactly all the realizations and calculate the MFPT explicitly for each of them employing (1). Thus we get a set $\{t_i; i = 1, 2^N\}$ of values of MFPT. For convenience, we rescale t_i by defining,

$$\tilde{t}_i = \frac{t_i - t_{\min}}{t_{\max} - t_{\min}} \tag{2}$$

where $t_{\max}$ and $t_{\min}$ denote the maximum and minimum of the values t respectively. $\{\tilde{t}_i\}$ can then be considered as dots on the real line segment (0,1). We investigate the nature of the density distribution of these dots employing multifractal formalisms [9, 10].

We divide the unit line segment into a set of 2^N non-overlapping intervals each of size, say $l (= 2^{-N})$. We define the partition function as,

$$Z(q, l) = \sum_{i=1}^{2^N} \rho_i^q(l) \tag{3}$$

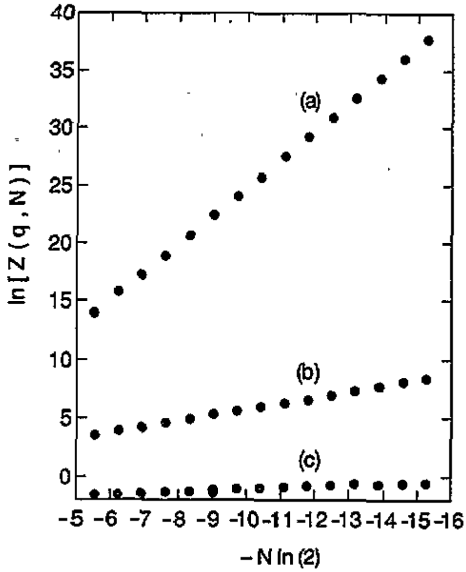

Figure 1. $\ln[Z(q,N)]$ versus $\ln[2^{-N}]$ for three values of q. (a) $q=-2$; (b) $q=0$; and (c) $q=2$.

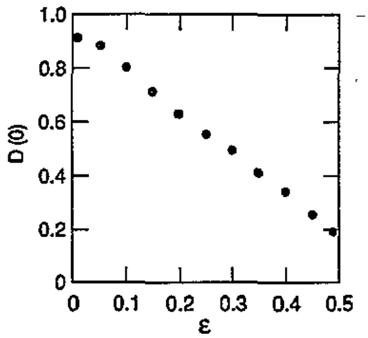

Figure 2. Variation of the fractal dimension with the disorder parameter ϵ. The data were obtained for $N=20$.

where $\rho_i(l)$ is the fractional number of dots in the ith interval and the sum runs over non-empty intervals only. We make the scaling ansatz that $Z(q,l) \sim l^{\tau(q)}$. The scaling exponents $\tau(q)$ are then given by

$$\tau(q) = \lim_{l\to 0} \frac{\ln[Z(q,l)]}{\ln[l]}. \tag{4}$$

In this expression the limit $l \to 0$ is obtained by letting the system size $N \to \infty$. The generalized Renyi dimensions are then given by $D(q) = \tau(q)/(q-1)$.

We carried out calculations with the lattice size N varying from 1 to 22. Figure 1 depicts the variation of $\ln[Z(q,N)]$ versus $-N\ln(2)$ for representative values of $q=-2, 0$, and $+2$, with $\epsilon = 0.25$. The linearity of the curve for large N establishes the scaling ansatz.

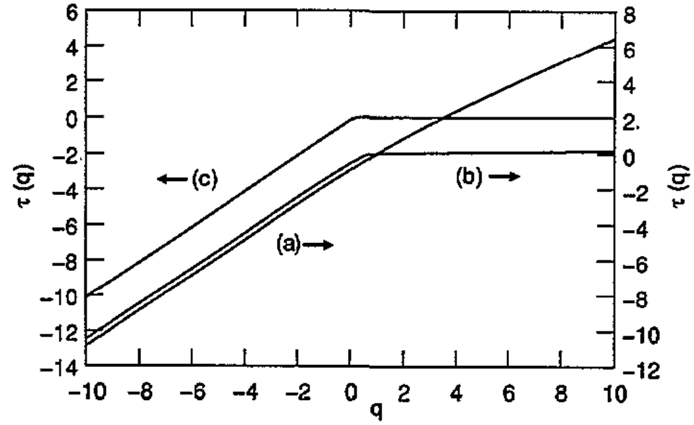

Figure 3. The scaling exponent τ versus q, for three values of the disorder parameter ϵ. (a) $\epsilon = 0.01$; (b) $\epsilon = 0.25$, and (c) $\epsilon = 0.49$. Note the shifted scales of the ordinates; curves (a) and (b) refer to the right scale, curve (c) to the left scale.

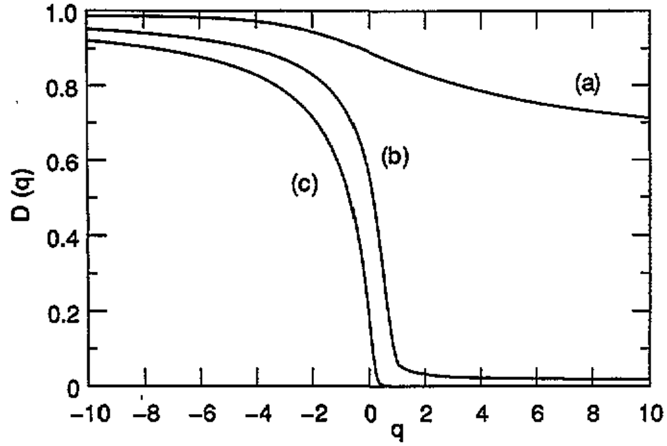

Figure 4. The $D(q)$ spectra of the distribution of MFPT for three values of the disorder parameter. (a) $\epsilon = 0.01$, (b) $\epsilon = 0.25$, and (c) $\epsilon = 0.49$.

Figure 2 depicts the fractal dimension given by $D(q = 0)$, calculated for various values of ϵ, in the range 0.01–0.49. We note that if $\epsilon \equiv 0.5$, then $t_i \equiv \infty$ for all i, except the one (say $i = \nu$) for which $t_\nu \equiv N$, since for such a lattice segment the right jump probabilities at all the sites are unity. Thus for $\epsilon \to 0.5$, we have $D(0) \to 0.0$. On the other hand if $\epsilon = 0$, there is no disorder; all the 2^N values of t would be identically the same, and equal to N^2. (For this situation the scaling described by (2) is not meaningful.) Hence $D(0) \equiv 0$. However, when ϵ is infinitesimally away from 0, even though all the values of t would be very close to each other, because of the scaling, see (2), $\tilde{t}$ would be more or less uniformly distributed in the unit line segment, and $D(0)$ would be nearly unity. For other values of ϵ, $D(0)$ lies between 0 and 1. The interesting observation is that the fractal dimension varies nearly linearly with ϵ as seen in figure 2.

The exponents $\tau(q)$ versus q for three representative values of $\epsilon = 0.01$, 0.25 and 0.49

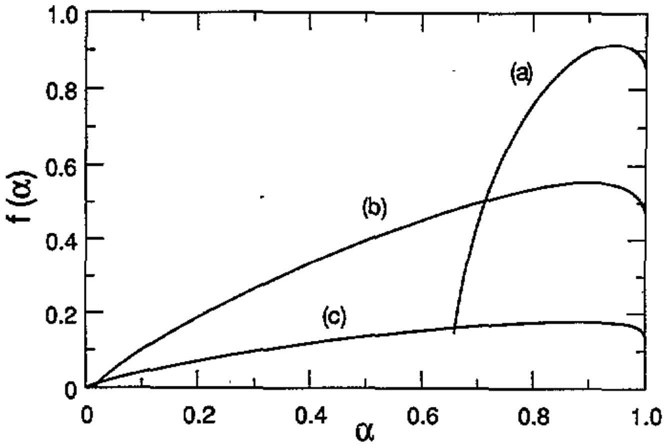

Figure 5. The $f(\alpha)$ spectrum for three values of the disorder parameter, ϵ. (a) $\epsilon = 0.01$, (b) $\epsilon = 0.25$, and (c) $\epsilon = 0.49$.

are depicted in figure 3. We see $\tau(q)$ is well defined exhibiting a change in slope. The generalized Renyi dimensions are shown in figure 4. Further for given $\tau(q)$, the spectrum of scaling indices usually denoted by $f(\alpha)$ can be obtained by Legendre transformation as given below,

$$f(\alpha) = \alpha q - \tau(q) \tag{5a}$$

$$\alpha = \frac{d}{dq}\tau(q). \tag{5b}$$

The $f(\alpha)$ spectra for three typical values of ϵ are shown in figure 5. Note the $f(\alpha)$ curve has a single hump shape and it depends on ϵ

In summary, we have studied the multifractal characteristics of the distribution of MFPT over Sinai disorder. We have calculated the fractal dimension $D(0)$ and found that it varies nearly linearly with increasing strength of disorder. We have obtained the exponents $\tau(q)$ that describe the scaling behaviour of the partition function and the generalized dimensions $D(q)$. We have also presented results on the $f(\alpha)$ spectrum.

One of the authors (SR) is thankful to Materials Science Division, Indira Gandhi Centre for Atomic Research for the award of a visiting scientist position and for its hospitality. KPNM is thankful to M C Valsakumar, G Ananthakrishna, R E Amritkar, and Neelima Gupte for useful discussions.

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